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General solutions of the Monge–Ampère equation in n-dimensional space

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Abstract

It is shown that the general solution of a homogeneous Monge–Ampère equation in n-dimensional space is closely connected with the exactly (but only implicitly) integrable system $\partial\xi_j/\partial x_0 + \sum_{k=1}^{n-1} \xi_k \partial\xi_j/\partial x_k = 0$ Using the explicit form of solution of this system it is possible to construct the general solution of the Monge–Ampère equation.

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1. Introduction

The original form of the Monge–Ampère equation in n-dimensional space is as follows [1–4]:

$$\det(\phi_{\rho,\sigma}) = F(\phi, \phi_\rho, x_\rho) \quad (0 \geq \rho, \sigma \geq n-1) \tag{1}$$

where $\phi_{\rho,\sigma} \equiv \partial^2\phi/\partial x_\rho \partial x_\sigma$ and x_ρ are the coordinates of the space.

With respect to each of its independent coordinates (1) is an equation of second order and consequently its general solution must depend on two arbitrary functions of $n-1$ independent arguments. We shall show that the homogeneous form of this equation, i.e., the special case where the function $F \equiv 0$, is exactly integrable. In this case the equation corresponds to the condition that a hypersurface in $\mathbb{R}^{n+1}$ has zero Gaussian Curvature. In what follows we shall refer to this equation with $F = 0$ as the Monge–Ampère equation. There is a deep connection between this equation and the equation

$$\det \begin{vmatrix} k\phi & \phi_j \\ \phi_i & \phi_{ij} \end{vmatrix} = 0. \tag{2}$$

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For all $k \neq 0$ this equation after a trivial change of unknown function ($\phi^{k/(k-1)}$ if $k \neq 1, 0$, $\log \phi$ if $k = 1$) coincides with the Monge–Ampère equation. The particular case $k = 0$ corresponds to the Universal Field Equation [8], whose general solution was found by means of a Legendre Transform in [7]. This solution is connected with the solution of the Monge–Ampère equation in $n+1$ dimensional space by the substitution

$$\Phi_{MA}(x_1, x_2, \ldots, x_{n+1}) = x_{n+1}\phi_{\text{UFE}}(x_1, x_2, \ldots, x_n) \tag{3}$$

Its solution by the methods of the present paper will be given elsewhere.

2. The general construction

The vanishing of the determinant means that the rows (or columns) of its $n \times n$ matrix are linearly dependent,

$$\sum_\sigma \phi_{\rho,\sigma}\alpha_\sigma = 0\,, \tag{4}$$

where α_σ are some functions of the coordinates x_ρ. From this observation it is easy to see that if ϕ is homogeneous of weight one, i.e. $\phi(\lambda x_\rho) = \lambda\phi(x_\rho)$, then by Euler's theorem on homogeneous functions,

$$\sum_\sigma x_\sigma \frac{\partial \phi}{\partial x_\sigma} - \phi = 0. \tag{5}$$

Differentiation of this equation with respect to x_ρ leads to an equation of the form (4) and thus implies that any homogeneous function of weight one satisfies the Monge–Ampère equation. Thus one class of solutions is easily obtained.

Let us rewrite (4) in the equivalent form

$$\frac{\partial \mathcal{R}}{\partial x_\rho} \equiv \frac{\partial}{\partial x_\rho}\sum_\sigma \phi_\sigma\alpha_\sigma = \sum_\sigma \phi_\sigma \frac{\partial \alpha_\sigma}{\partial x_\rho} \tag{6}$$

and will consider $\mathcal{R}$ as a function of n independent variables α_ρ assuming that $\det(\partial\alpha_\sigma/\partial x_\rho) \neq 0)$.

From (6) we obtain immediately

$$\phi_\sigma = \frac{\partial \mathcal{R}}{\partial \alpha_\sigma}, \quad \mathcal{R} = \sum \alpha_\sigma \frac{\partial \mathcal{R}}{\partial \alpha_\sigma}$$

or that $\mathcal{R}$ is a homogeneous function of the variables α_ρ of degree one. So

$$\mathcal{R} = \alpha_0 R(\alpha_j/\alpha_0), \quad \phi_0 = R - \sum \xi_j \frac{\partial R}{\partial \xi_j}, \quad \phi_j = \frac{\partial R}{\partial \xi_j}. \tag{7}$$

Here $\xi_j = \alpha_j/\alpha_0$ and from now on all Latin indices take values from 1 up to $n-1$.

The condition of compatibility of (7) – the equivalence of the second mixed derivatives – will give us the dependence of the $n-1$ new variables ξ_j upon the n previous variables x_ρ.

We have firstly

$$\frac{\partial}{\partial x_k}\frac{\partial R}{\partial \xi_j} = \frac{\partial}{\partial x_j}\frac{\partial R}{\partial \xi_k} \tag{8}$$

and secondly

$$\frac{\partial}{\partial x_k}\frac{\partial \phi}{\partial x_0} = -\sum \xi_j \frac{\partial}{\partial x_k}\frac{\partial R}{\partial \xi_j} = -\sum \xi_j \frac{\partial}{\partial x_j}\frac{\partial R}{\partial \xi_k} = \frac{\partial}{\partial x_0}\frac{\partial R}{\partial \xi_k}. \tag{9}$$

From the last equality (9) we readily obtain

$$\sum_k \left(\frac{\partial \xi_k}{\partial x_0} + \sum_{r=1}^{n-1} \xi_r \frac{\partial \xi_k}{\partial x_r} \right) \frac{\partial^2 R}{\partial \xi_k \partial \xi_j} = 0. \tag{10}$$

With respect to the variables

$$q_k = \frac{\partial \xi_k}{\partial x_0} + \sum_{r=1}^{n-1} \xi_r \frac{\partial \xi_k}{\partial x_r}$$

Eq. (10) is a linear system of algebraic equations which it is possible to solve in the two cases given by the Fredholm alternative:

$$\det_{n-1}\left(\frac{\partial^2 R}{\partial \xi_k \partial \xi_j}\right) = 0, \quad q_k \neq 0 \quad \text{for some } k,$$

$$\det_{n-1}\left(\frac{\partial^2 R}{\partial \xi_k \partial \xi_j}\right) \neq 0, \quad q_k = 0. \tag{11}$$

We shall consider in this paper the second possibility, hoping to come back to the first one in further publications and restrict ourselves now only to the simplest nontrivial example $n = 3$ in Section 5.

3. Solution of equations of hydrodynamic type

From the physical point of view, the system of equations

$$\frac{\partial \xi_k}{\partial x_0} + \sum_{k=1}^{n-1} \xi_k \frac{\partial \xi_k}{\partial x_k} = 0 \tag{12}$$

is that of the equations of velocity flow in hydrodynamics in $n-1$ dimensional space.

In the simplest case $n = 2$ this is the one component equation

$$\frac{\partial \xi}{\partial x_0} + \xi \frac{\partial \xi}{\partial x_1} = 0, \tag{13}$$

which is connected with the name of Monge, who first discovered its general solution in implicit form,

$$x_1 - \xi x_0 = f(\xi),$$

where $f(\xi)$ is an arbitrary function of its argument.

The generalisation of this result to the multidimensional case (12) was found by D.B. Fairlie [8] and consists in the following construction. Suppose that the system of $n-1$ equations

$$x_j - \xi_j x_0 = Q^j(\xi_1, \xi_2, \dots \xi_{n-1}) \tag{14}$$

is solved for the unknown functions ξ_j. Each solution of this system (14) will satisfy (12).

This may be proved as follows. After differentiation of each equation of (14) with respect to the variables x_k we obtain

$$\delta_{k,j} = \sum \left(\frac{\partial Q^j}{\partial x_r} + x_0 \delta_{j,r} \right) \frac{\partial \xi_r}{\partial x_k} \tag{15}$$

or

$$\frac{\partial \xi_j}{\partial x_k} = \left(x_0 \mathbb{I} + \frac{\partial Q}{\partial \xi} \right)^{-1}_{j,k} . \tag{16}$$

Here $\partial Q / \partial \xi$ denotes a matrix whose (j,k)th element is $\partial Q^j / \partial \xi_k$. The result of differentiation of (14) with respect to x_0 has as a corollary

$$-\frac{\partial \xi_j}{\partial x_0} = \left[\left(x_0 \mathbb{I} + \frac{\partial Q}{\partial \xi} \right)^{-1} \xi \right]_j . \tag{17}$$

The comparison of (16) and (17) proves that the system (12) is satisfied.

4. Continuation

Up to now we have used only the condition of compatibility (9). Now let us return to Eqs. (8). By simple computations using the explicit form for derivatives (16) it is easy to show that Eqs. (8) may be written in the form

$$\sum \frac{\partial}{\partial \xi_k} Q^r \frac{\partial^2 R}{\partial \xi_r \partial \xi_j} = \sum \frac{\partial}{\partial \xi_j} Q^r \frac{\partial^2 R}{\partial \xi_r \partial \xi_k} \tag{18}$$

or

$$\sum \frac{\partial^2 R}{\partial \xi_r \partial \xi_j} Q^j = \frac{\partial L}{\partial \xi_r} , \tag{19}$$

where L is an arbitrary function of $n-1$ arguments ξ_k. With respect to the functions Q^j Eqs. (19) form a linear system of algebraic equations, the matrix of which has determinant not equal to zero (the second case of (12)).

The solution of (19) in terms of Cramer's determinants has the usual form

$$Q^j = \left[\left(\frac{\partial^2 R}{\partial \xi \partial \xi} \right)^{-1} \frac{\partial L}{\partial \xi} \right]_j \tag{20}$$

and so all of the functions Q^j are represented in terms of only two arbitrary functions R, L and their derivatives up to the second and first orders, respectively.

So with the help of Eqs. (7), whose integrability is ensured by the above analysis, and (14) we obtain the solution of the Monge–Ampère equation, which depends upon two arbitrary functions each of $n-1$ independent arguments and which fulfils the claim of the introduction to provide the general solution for the equation under consideration.

5. The simplest examples

5.1. $n=2$

From (7) we obtain

$$\frac{\partial\phi}{\partial x_0} = R - \xi_1\frac{\partial R}{\partial \xi_1}, \qquad \frac{\partial\phi}{\partial x_1} = \frac{\partial R}{\partial \xi_1}. \tag{21}$$

Eq. (13) and its solution are as follows:

$$\frac{\partial\xi_1}{\partial x_0} + \xi_1\frac{\partial \xi_1}{\partial x_1} = 0, \quad x_1 - \xi_1 x_0 = F(\xi_1). \tag{22}$$

In this case the conditions (7) are absent and a general solution of the Monge–Ampère equation is determined by the pair of arbitrary functions R, F each of one argument ξ_1.

5.2. $n=3$ – the general case

From (7) we obtain

$$\frac{\partial\phi}{\partial x_0} = R - \xi_1\frac{\partial R}{\partial \xi_1} - \xi_2\frac{\partial R}{\partial \xi_2}, \quad \frac{\partial\phi}{\partial x_1} = \frac{\partial R}{\partial \xi_1}, \quad \frac{\partial\phi}{\partial x_2} = \frac{\partial R}{\partial \xi_2}. \tag{23}$$

The connection between the arguments ξ and x keeping in mind (20) takes the form

$$\begin{aligned} x_1 - \xi_1 x_0 = Q^1 &= \frac{R_{22}L_1 - R_{12}L_2}{R_{22}R_{11} - R_{12}R_{21}}, \\ x_2 - \xi_2 x_0 = Q^2 &= \frac{-R_{12}L_1 + R_{22}L_2}{R_{22}R_{11} - R_{12}R_{21}}, \end{aligned} \tag{24}$$

where $L_1 \equiv \partial L/\partial\xi_1$ and so on.

5.3. $n=3$ – the degenerate case

The first possibility in (12) means $\mathrm{Det}_2(\partial^2 R/\partial\xi\,\partial\xi) = 0$. This is precisely the Monge–Ampère equation of the first subsection with the solution

$$\frac{\partial R}{\partial \xi_2} = P - q\frac{\partial P}{\partial q}, \quad \frac{\partial R}{\partial \xi_1} = \frac{\partial P}{\partial q}, \quad \xi_1 - q\xi_1 = F(q).$$

Let us substitute all these expressions into (23). We obtain

$$\frac{\partial\phi}{\partial x_0} = R - (\xi_1 - \xi_2 q)\frac{\partial P}{\partial q} - \xi_2 P, \quad \frac{\partial\phi}{\partial x_2} = P - q\frac{\partial P}{\partial q}, \quad \frac{\partial\phi}{\partial x_1} = \frac{\partial P}{\partial q}.$$

The condition of compatibility (the equality of second mixed derivatives) gives the equations

$$q_{x_2} = -q q_{x_1}, \quad q_{x_0} = -F(q) q_{x_1}, \quad q q_{x_3} = -F(q) q_{x_2},$$

which are compatible and have the common solution

$$-x_1 + q x_2 - F(q) x_0 = \Phi(q),$$

where $\Phi(q)$ is an arbitrary function of its argument.

We see that the degenerate solution of the Monge–Ampère equation is determined in this case by three arbitrary functions of one argument.

6. Concluding remarks

The general solution of the the Monge–Ampère equation constructed here reminds one of the situation with regard to linear ordinary second order equations when one solution is known; there is a well known procedure for the construction of a first order differential equation whose solution yields the general solution of the original equation. Here something of the same nature has been discovered; an arbitrary homogeneous function of weight one satisfies the Monge–Ampère equation automatically. We then construct a set of first order equations which are integrable, though only implicitly. Their solution then implies the general solution to our original equation. This construction suggests that equations with the property that any homogeneous function of given degree is a solution may be also solved by this method, for example the Universal field equation introduced in [5,6], whose three dimensional version describes developable surfaces and which admits homogeneous solutions of degree zero. We intend to return to this matter in a subsequent paper.

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